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Bifurcations of a Pair of Nonorientable Heteroclinic Cycles

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In this paper we study the bifurcations of a pair of nonorientable heteroclinic cycles. In addition to the obvious and important bifurcation "Ω-explosion," several other bifurcations, for example, homoclinic and heteroclinic bifurcation behaviors, are described in terms of symbolic sequences and symbolic descriptions of trajectories staying forever in a sufficiently small neighborhood of the cycles are established.

Key Words: nonorientable heteroclinic cycles; symbol sequence; product space.

1. INTRODUCTION

In this paper we study some bifurcation problems of a pair of nonorientable heteroclinic cycles of vector fields, which are related to the study of Lorenz equations. For example, in [1] it is observed by computer simulation of the Lorenz equations that some heteroclinic cycle structure causes turbulent behavior of the trajectories of this system and it is believed that this cycle structure of vector fields causes a transition to turbulence. The study of this cycle structure is divided into several cases: both cycles are orientable; one orientable, the other nonorientable; both cycles are nonorientable. In [2, 3], the case that both cycles are orientable

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is studied, and it is proved that there are infinitely many trajectories which stay forever in a sufficiently small neighborhood of the contour of this pair of orientable heteroclinic cycles and can be described by symbolic systems of finite or countably infinite symbols. When the system which possesses this pair of cycles is perturbed by a generic two-parameter system, it is also shown how the symbolic descriptions of the trajectories which stay forever in a sufficiently small neighborhood of the contour of the cycles change with parameters and infinitely many bifurcation curves are found.

The situation of just one cycle being orientable is radically different from the first case in the symbolic descriptions of trajectories staying forever in the neighborhood of the contour and as a result the bifurcation behaviors are also very different. The detailed discussion of this case is given in [4].

We can use a few words to say about the whole situation, that is, for the first two cases, when parameters vary a little, no obvious Ω-explosion is easily observed, where we mean there is no "sudden" increase of the set of nonwandering points in a neighborhood of this pair of cycles, and when computer simulation is made, a chaotic invariant set seems always to exist. However, for the case that both cycles are nonorientable, it is certain that obvious Ω-explosion occurs, that is, for some parameters, the set of nonwandering points of the system in the neighborhood of the pair of cycles consists of just one saddle point and two hyperbolic periodic orbits, while for some other parameters, the set of nonwandering points of the system in the neighborhood of the cycles contains infinitely many trajectories. In this paper we will show what kinds of bifurcation behavior happen for a generic two-parameter unfolding of a system with a pair of nonorientable heteroclinic cycles and give symbolic descriptions about the trajectories which stay forever in a sufficiently small neighborhood of the pair of cycles.

Now we state the problem precisely. We consider the two-parameter system E_ε,

$$\frac{du}{dt} = E(u, \varepsilon), \tag{1}$$

where $u \in R^3$, $\varepsilon \in D \subset R^2$, D is a neighborhood of the origin, $E(0, \varepsilon) = 0$, $E(u, \varepsilon) \in C^r$ $(r \geq 3)$. The eigenvalues of the linearization of $E(u, 0)$ at the origin are λ_1, λ_2, γ, with $\lambda_2 < \lambda_1 < 0 < \gamma$, $\lambda_1 + \gamma > 0$.

Hence, for $\varepsilon = 0$, the stable manifold W_0^s of the origin is two-dimensional, the unstable manifold $W_0^u = \Gamma_1 \cup \Gamma_2 \cup \{0\}$ is one-dimensional. The strong stable manifold W_0^{ss} in W_0^s divides W_0^s into two parts W_0^+ and W_0^- in the neighborhood of the origin.

In addition we suppose E_0 has two hyperbolic periodic orbits L_α ($\alpha = 1, 2$) with two-dimensional stable manifold W_α^s, two-dimensional unstable manifold W_α^u, $\Gamma_\alpha \subset W_\alpha^s$ for $\alpha = 1, 2$, and W_α^u intersects transversally with W_0^+ (or W_0^-) along a trajectory γ_α, $\alpha = 1, 2$, see Fig. 1. In this way the system E_0 has a pair of heteroclinic cycles $\Gamma_\alpha \cup L_\alpha \cup \gamma_\alpha \cup \{0\}$, $\alpha = 1, 2$. In this paper we assume this pair of cycles is nonorientable. The precise meaning of the cycles being nonorientable is given in Section 2. At the present stage, we refer to Fig. 2.

We will study what kinds of bifurcation behavior happen for a "generic" unfolding E_ε of E_0, where "generic" means that for the system E_ε, the heteroclinic orbit Γ_α ($\alpha = 1, 2$) occurs and breaks in a transversal manner with the variation of parameters (for the precise meaning of this condition, see (6) of Section 2).

In order to simplify the process of argumentation, we assume that

(A1) the eigenvalues of the linearization of the vector field E_0 at the origin are nonresonant;

(A2) the multipliers a_α, b_α, of L_α are nonresonant with $0 < a_\alpha < 1 < b_\alpha$.

To assure the problem we study here is of codimension two, we further assume

(A3) the two-dimensional invariant manifold W which is tangent to the plane spanned by the eigenvectors associated to λ_1 and γ at the origin intersects transversally with W_α^s along Γ_α, $\alpha = 1, 2$.

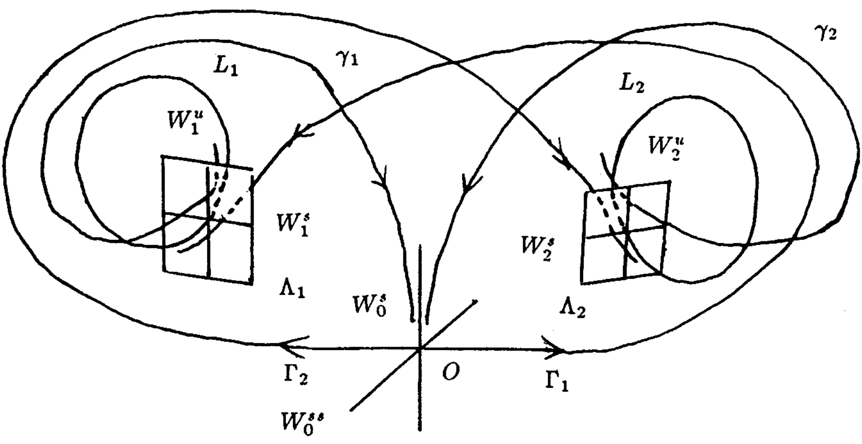

FIG. 1. Heteroclinic cycles. Reprinted from *Nonlinear Anal. TMA* **29** (1997), Qi Dongwen, Bifurcations of a pair of singular cycles, pp. 314–315, with permission from Elsevier Science.

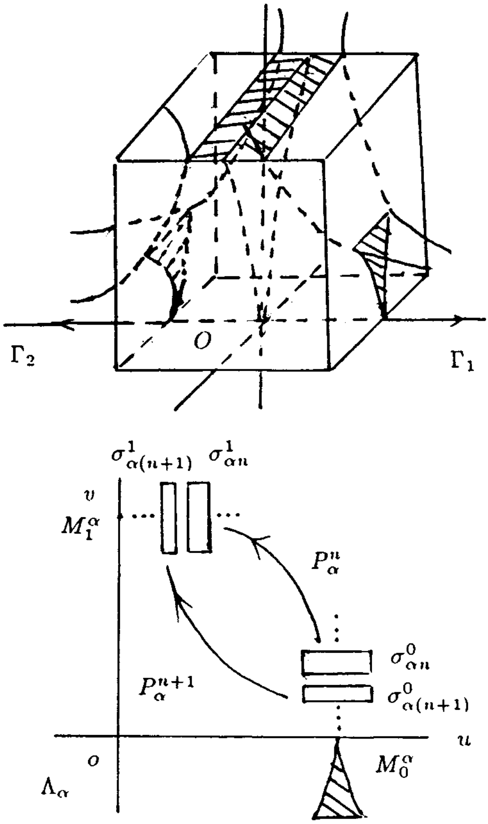

FIG. 2. Nonorientable case, $\alpha = 1, 2$. Reprinted from *Nonlinear Anal. TMA* **29** (1997), Qi Dongwen, Bifurcations of a pair of singular cycles, pp. 314–315, with permission from Elsevier Science.

For the existence of the invariant manifold W, see [5]. Clearly this invariant manifold contains the unstable manifold of the origin. And it is not unique. However, it has the unique tangent space at any point on the unstable manifold of the origin. For related statements, see [6, 7] and a monograph by Yu. Il'yashenko, Li Weigu, and co-workers [8] on normal forms and nonlocal bifurcations.

We should mention the paper [9] written by Bamón *et al.*, where the bifurcations of one heteroclinic cycles are studied in great detail, the properties of the set of bifurcation parameters are studied and the measure of the set of bifurcation parameters is estimated. They show that the parameter region is largely filled by Axiom A flows and non-Axiom A flows are arranged in a codimension one lamination of the space of vector fields. Their approach is to reduce the two-dimensional map to a one-dimensional, possibly multiply defined, map. This reduction is obtained by proving that there exists a strong stable foliation for the flow, or, alternatively a stable foliation for the return map. This kind of approach can also be used in the study of our question and some results similar to those given in [9] can be obtained with a little difficulty, where we have to consider the influences of the simultaneous occurrence of two cycles. However, the way we attack the question is more elementary, although it is also complicated, and some geometric descriptions can be given in the process of discussions.

The bifurcation problem we study here is of codimension two, in the sense of the simultaneous occurrence of two heteroclinic cycles and influences of each other during the unfolding of the system E_0. There are several other articles devoted to the study of some codimension two bifurcations generating horseshoes or attractors. For example, see [6, 10–12], where bifurcations usually happen as the result of some codimension one degeneracy of a homoclinic orbit. In our paper, although the measure of the bifurcation parameters cannot be estimated, the types of bifurcation curves are found and described in terms of some symbol sequences, the correspondence between trajectories staying forever in a neighborhood of the contour of the pair of cycles and some symbol sequences is also established, and the great complexity of bifurcation behaviors is shown.

Now to describe the question precisely, we make some preparations.

2. POINCARÉ MAPS BETWEEN CROSS SECTIONS

2.1. *Cases in the Neighborhood of the Origin*

Since the eigenvalues of the linearization of the vector field E_0 at the origin are nonresonant, by the normal form theory of local families, for example, see [8] or [12], we can choose coordinates (x, y, z) for sufficiently small ε, in some neighborhood V of the origin, such that

(1) $\{(x, y, z) \mid |x| \le 1, |y| \le 1, |z| \le 1\} \subset V$,

(2) the system is linear in V,

$$\begin{aligned} \dot{x} &= \lambda_1(\varepsilon)x, \\ \dot{y} &= \lambda_2(\varepsilon)y, \\ \dot{z} &= \gamma(\varepsilon)z, \end{aligned} \tag{2}$$

where $\lambda_i(0) = \lambda_i$, $i = 1, 2$, $\gamma(0) = \gamma$.

Choose the following cross sections of the vector field,

$$S_\alpha = \{(x, y, z) \mid |x| \le 1, |y| \le 1, z = (-1)^{\alpha-1}\},$$

$$S_0^\pm = \{(x, y, z) \mid x = \pm 1, |y| \le 1, |z| \le 1\},$$

$$D_h^1 = \{(x, y, z) \mid x = 1, |y| \le 1, 0 < z < h\},$$

$$D_h^2 = \{(x, y, z) \mid x = 1, |y| \le 1, -h < z < 0\},$$

and assume $\gamma_\alpha \cap S_0^+ = \{(1, y_\alpha, 0)\}$, where $y_\alpha \in (-1, 1)$.

The correspondence map T_α: $D_h^\alpha \to S_\alpha$ along trajectories of the system E_ε can be written as

$$T_\alpha(y, z) = \left(|z|^{\theta_1(\varepsilon)}, y|z|^{\theta_2(\varepsilon)}\right) = \phi_\alpha(y, z, \varepsilon)|z|^{\theta_1}, \tag{3}$$

with $\theta_1(\varepsilon) = -\lambda_1(\varepsilon)/\gamma(\varepsilon) < \theta_2(\varepsilon) = -\lambda_2(\varepsilon)/\gamma(\varepsilon)$, $\phi_\alpha(y, z, \varepsilon) = (1, y|z|^{\theta_2 - \theta_1})$, where $\theta_1(\varepsilon) < 1$. We have the following lemma.

LEMMA 2.1. (a) *There exist constants M and $\bar{e}$ $(0 < \bar{e} < 1)$ such that*

$$\left\| \frac{\partial \phi_\alpha}{\partial y} \right\| < M, \qquad \left\| \frac{\partial \phi_\alpha}{\partial z} \right\| |z|^{\bar{e}} < M.$$

(b) $\lim_{z \to 0} \phi_\alpha(y, z, \varepsilon) = \binom{1}{0}$.

Remark 2.1. If the eigenvalues of the saddle point do not satisfy the nonresonance condition, this kind of lemma can also be obtained by using Deng's method on so-called exponential expansions [13]. For details, see [2].

2.2. *Poincaré Maps along Periodic Orbits*

For $|\varepsilon| \ll 1$, the system E_ε possesses a periodic orbit $\tilde{L}_\alpha = L_\alpha(\varepsilon)$ which is the continuation of L_α, $\tilde{L}_\alpha \to L_\alpha$ when $\varepsilon \to (0,0)$. Choose a cross section Λ_α at a point of $L_\alpha(\varepsilon)$. Because of the nonresonance condition (A2), by the normal form theory of local families [8, 12], we can choose coordinates (u, v) in some neighborhood U_α of $\tilde{L}_\alpha \cap \Lambda_\alpha$ on Λ_α such that

(1) $\tilde{L}_\alpha \cap \Lambda_\alpha = \{(u, v) = (0, 0)\}$;

(2) $\{(\mathrm{u}, \mathrm{v}) \mid |\mathrm{u}|, |\mathrm{v}| \le 2\} \subset \mathrm{U}_\alpha$;

(3) $\{(u, 0) \mid |u| \le 2\} \subset W_\alpha^s$;

(4) $\{(0, \mathrm{v}) \mid |\mathrm{v}| \le 2\} \subset \mathrm{W}_\alpha^{\mathrm{u}}$;

(5) the Poincaré map P_α defined on U_α has the form

$$(u, v) \mapsto (a_\alpha(\varepsilon)u, b_\alpha(\varepsilon)v), \tag{4}$$

with $0 < a_\alpha(\varepsilon) < q_1 < 1 < q_2 < b_\alpha(\varepsilon)$, q_1, q_2 being fixed constants. We assume that there exist $u_\alpha^+, v_\alpha^- \in (0,1)$ such that the points $M_0^\alpha := (u_\alpha^+, 0)$ and $M_1^\alpha := (0, v_\alpha^-)$ satisfy $M_0^\alpha \in \Gamma_\alpha \cap U_\alpha$, $M_1^\alpha \in \gamma_\alpha \cap U_\alpha$. Let $\Lambda_\alpha^0 = \{(u,v) \in U_\alpha \mid |u - u_\alpha^+| \le \delta, |v| \le \delta\}$, $\Lambda_\alpha^1 = \{(u,v) \in U_\alpha \mid |u| \le \delta, |v - v_\alpha^-| \le \delta\}$. If δ is sufficiently small, we have $P_\alpha \Lambda_\alpha^0 \cap \Lambda_\alpha^0 = \emptyset$, $P_\alpha^{-1}\Lambda_\alpha^1 \cap \Lambda_\alpha^1 = \emptyset$, $0 < u_\alpha^+ - \delta < u_\alpha^+ + \delta < 1$, $0 < v_\alpha^- - \delta < v_\alpha^- + \delta < 1$.

For later applications, the map P_α^n: $U_\alpha \to U_\alpha, (u,v) \mapsto (\bar{u}, \bar{v}) = (a_\alpha^n(\varepsilon)u, b_\alpha^n(\varepsilon)v)$ is expressed in another way as $(u, \bar{v}) \mapsto (\bar{u}, v) = (a_\alpha^n(\varepsilon)u, b_\alpha^{-n}(\varepsilon)\bar{v})$.

For sufficiently large n, the domain $\sigma_{\alpha n}^0$ of definition of the map P_α^n: $\Lambda_\alpha^0 \to \Lambda_\alpha^1$ is

$$\sigma_{\alpha n}^0 = \left\{(u,v) \mid |u - u_\alpha^+| \le \delta, b_\alpha^{-n}(\varepsilon)(v_\alpha^- - \delta) \le v \le b_\alpha^{-n}(\varepsilon)(v_\alpha^- + \delta)\right\},$$

and the range $\sigma_{\alpha n}^1$ is

$$\sigma_{\alpha n}^1 = \left\{(u,v) \mid a_\alpha^n(\varepsilon)(u_\alpha^+ - \delta) \le u \le a_\alpha^n(\varepsilon)(u_\alpha^+ + \delta), |v - v_\alpha^-| \le \delta\right\}.$$

2.3. *Poincaré Maps between Cross Sections in the Neighborhood of the Origin and Cross Sections of the Periodic Orbits*

Since for $\varepsilon = (0,0)$, Γ_α intersects with S_α and U_α at the points $M_{0\alpha}(0,0,(-1)^{\alpha-1})$ and $M_0^\alpha(u_\alpha^+, 0)$, respectively, it follows from the continuous dependence of solutions of differential equations upon initial conditions and parameters that for ε sufficiently small, the positive half trajectory which passes through any point in the neighborhood of $M_{0\alpha}$ in S_α intersects with U_α at a point near M_0^α. We denote this correspondence by $T_{1\alpha}$, which is expressed as

$$T_{1\alpha}\colon \quad \begin{aligned} u &= u_\alpha^+ + u_\alpha(\varepsilon) + J_\alpha(\varepsilon)\begin{pmatrix} x \\ y \end{pmatrix} + \cdots = U_\alpha(x,y,\varepsilon), \\ v &= v_\alpha(\varepsilon) + K_\alpha(\varepsilon)\begin{pmatrix} x \\ y \end{pmatrix} + \cdots = V_\alpha(x,y,\varepsilon), \end{aligned} \tag{5}$$

with $J_\alpha(\varepsilon)$, $K_\alpha(\varepsilon)$ being 1×2 matrices, and the dots denoting the higher-order terms of $\binom{x}{y}$, $u_\alpha(0) = v_\alpha(0) = 0$.

Now the precise meaning of the "genericity" of the unfolding E_ε of E_0 is that

$$\left.\begin{vmatrix} \dfrac{\partial v_1}{\partial \varepsilon_1} & \dfrac{\partial v_1}{\partial \varepsilon_2} \\ \dfrac{\partial v_2}{\partial \varepsilon_1} & \dfrac{\partial v_2}{\partial \varepsilon_2} \end{vmatrix}\right|_{(0,0,0)} \neq 0. \tag{6}$$

Therefore, we can use new parameters $\mu = (\mu_1, \mu_2)$, where $\mu_1 = v_1(\varepsilon)$, $\mu_2 = v_2(\varepsilon)$, to replace the parameters $\varepsilon = (\varepsilon_1, \varepsilon_2)$.

The transversality condition (A3) is equivalent to

$$C_\alpha = K_\alpha(0)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \neq 0. \tag{7}$$

The case $C_\alpha > 0$, $\alpha = 1, 2$, corresponds to the situation that both heteroclinic cycles are orientable, which is studied in [2, 3], while the case that $C_1 > 0$, $C_2 < 0$ corresponds to the situation that one cycle is orientable and the other is nonorientable, which is studied in [4]. In this paper we will give the bifurcation analysis of the case that $C_\alpha < 0$, $\alpha = 1, 2$, which corresponds to the situation that both cycles are nonorientable. See Fig. 2.

Finally, since W_α^u intersects transversally with W_0^+ along the trajectory γ_α, we have the correspondence map $T_{2\alpha}\colon \Lambda_\alpha^1 \to S_0^+$ determined by the flow of the vector field E_μ, which is expressed as $T_{2\alpha}\colon (u, v) \mapsto (y, z) = (f_\alpha(u, v, \mu), g_\alpha(u, v, \mu))$ and the transversality of the intersection of the invariant manifold W_α^u with W_α^+ implies that

$$\frac{\partial g_\alpha}{\partial v}(0, v_\alpha^-, 0) \neq 0. \tag{8}$$

We have described the correspondence map in the neighborhood of the saddle point (the origin), the correspondence maps between the neighborhood of the saddle point and the neighborhood of the periodic orbits, and the correspondence maps along periodic orbits. Now it is possible for us to use these maps, in the following sections, to study those trajectories which stay forever in a small neighborhood of the contour of the heteroclinic cycles.

3. SYMBOLIC SYSTEMS AND STATEMENTS OF THEOREMS

If a neighborhood $\tilde{N}$ of the contour of the heteroclinic cycles $\Gamma_1 \cup L_1 \cup \gamma_1 \cup \{0\} \cup \Gamma_2 \cup L_2 \cup \gamma_2$ is chosen to be sufficiently close to the cycles, then any trajectory except one of the pair of the cycles which stays forever in $\tilde{N}$ must run around any one of the periodic orbits L_α for more than (or equal to) $\tilde{k}$ times, where $\tilde{k}$ is a sufficiently large natural number, and when $\tilde{N}$ tends to the contour, $\tilde{k} \to +\infty$.

To state the theorem clearly, we need the following symbolic systems of bi-infinite sequences:

$$\Omega_\beta^- = \Big\{\big(\ldots,(\beta,\infty),\ldots,(\beta,\infty),(\alpha_j,k_j),\ldots,(\alpha_{j+i},k_{j+i}),\ldots\big)\Big\}$$

$$\tilde{\Omega} = \{(\ldots,(\alpha_{-i},k_{-i}),\ldots,(\alpha_0,k_0),\ldots,(\alpha_i,k_i),\ldots)\}$$

$$\Omega_\alpha^+ = \Big\{\big(\ldots,(\alpha_{j-i},k_{j-i}),\ldots,(\alpha_{j-1},k_{j-1}),(\alpha_j,k_j), (\alpha,\infty),\ldots,(\alpha,\infty),\ldots\big)\Big\}$$

$$\Omega_{\beta\alpha} = \Big\{\big(\ldots,(\beta,\infty),(\alpha_{j+1},k_{j+1}),\ldots,(\alpha_{j+l},k_{j+l}), (\alpha,\infty),\ldots,(\alpha,\infty),\ldots\big)\Big\}$$

$$\tilde{L}_\alpha = \{(\ldots,(\alpha,\infty),\ldots,(\alpha,\infty),\ldots)\}$$

$$W = \bigcup_{\alpha,\beta}\Big(\Omega_\beta^- \cup \Omega_\alpha^+ \cup \Omega_{\beta\alpha} \cup \tilde{\Omega} \cup \tilde{L}_\alpha\Big)$$

$$W_k^l = \Big\{\big(\ldots,(\tilde{\alpha}_i,\tilde{k}_i),\ldots\big) \in W \,|\, \tilde{k}_i \ge k, \text{ if } \tilde{\alpha}_i = 1;\ \tilde{k}_i \ge l, \text{ if } \tilde{\alpha}_i = 2\Big\}$$

$$\tilde{W} = \Omega_1^- \cup \Omega_2^- \cup \tilde{\Omega} \cup \Omega_0^+ \cup \Omega_{10} \cup \Omega_{20} \cup \tilde{L}_1 \cup \tilde{L}_2 \cup \tilde{L}_0$$

$$W(\beta) = \tilde{W} \cup \Omega_\beta^+ \cup \Omega_{1\beta} \cup \Omega_{2\beta}$$

$$W(0) = \tilde{L}_0 \cup \tilde{L}_1 \cup \tilde{L}_2$$

$$W(\beta,k) = \Big\{\big(\ldots,(\tilde{\alpha}_i,\tilde{k}_i),\ldots\big) \in W(\beta) \,|\, \tilde{k}_i \ge k, \text{ if } \tilde{\alpha}_i = \beta;\ \tilde{k}_i = \infty, \text{ if } \tilde{\alpha}_i \ne \beta\Big\};$$

here $\beta = 1,2$; $\alpha = 0,1,2$; $\alpha_i = 1,2$; $k_i \in \mathbf{N}$ (the set of natural numbers), and $k_i \ge \tilde{k}$; $l \in \mathbf{N}$, $j \in \mathbf{Z}$ (the set of integers).

For the symbolic system W and its subsystems, a shift operator is defined by σ: $\sigma a = a'$, where $a = (\ldots,(\alpha_i,k_i),\ldots)$, $a' = (\ldots,(\alpha_i',k_i'),\ldots)$, $(\alpha_i',k_i') = (\alpha_{i+1},k_{i+1})$. The set W is given the usual topology, that is, we define a distance d on W. Let

$$|(\alpha,k)-(\alpha',k')| = \begin{cases} 1, & \text{if } (\alpha,k) \ne (\alpha',k'), \\ 0, & \text{if } (\alpha,k) = (\alpha',k'), \end{cases}$$

and

$$d(a,b) = \sum_{i=-\infty}^{\infty} |(\alpha_i,k_i)-(\beta_i,l_i)|/2^{|i|},$$

where $a = (\dots,(\alpha_i, k_i),\dots)$, $b = (\dots,(\beta_i, l_i),\dots)$. Now the symbolic system (W, σ) and its various subsystems are very useful for us to describe trajectories which stay forever in the neighborhood $\tilde{N}$ of the contour of the pair of cycles.

We make some explanations about this correspondence. Suppose a trajectory γ of the system E_ε stays forever in the neighborhood $\tilde{N}$ of the contour and does not tend to any one of the three critical elements $L_\alpha(\varepsilon)$, $\alpha = 0, 1, 2$, then it must intersect successively with the cross section Λ_{α_i} at a point in the region $\sigma_{\alpha_i k_i}$, running along the periodic orbit $L_{\alpha_i}(\varepsilon)$ for k_i times and then going to the cross section $\Lambda_{\alpha_{i+1}}$ and so on. Therefore the trajectory γ corresponds with the symbol sequence

$$(\dots,(\alpha_{-1}, k_{-1}),(\alpha_0, k_0),(\alpha_1, k_1),\dots).$$

If a trajectory finally tends to the critical element $L_\alpha(\varepsilon)$, we make a little change by setting $(\alpha_i, k_i) = (\alpha, \infty)$ for all sufficiently large positive i, etc. In Section 5, we will prove that these correspondences are one to one by using some contraction mapping theorem.

The fundamental theorems of this paper are as follows.

THEOREM 3.1. *If E_ε is a generic unfolding of the system E_0, which has a pair of nonorientable heteroclinic cycles $\Gamma_1 \cup L_1 \cup \gamma_1 \cup \{0\} \cup \Gamma_2 \cup L_2 \cup \gamma_2$ and satisfies the conditions* (A1)–(A3), *then there exists a neighborhood $\tilde{N}$ of the contour of the pair of cycles, a neighborhood V of the origin in the $(\varepsilon_1, \varepsilon_2)$ parameter plane, and a parameter transformation $\mu = \mu(\varepsilon) = (\mu_1, \mu_2)$ on V with $\mu(0,0) = (0,0)$ such that the following conclusions hold.*

There exist C^1-functions $f_\alpha^-(\mu_\beta, k), f_\alpha^+(\mu_\beta, k) (k \geq \tilde{k})$ satisfying

$$f_\alpha^-(\mu_\beta, k) < f_\alpha^+(\mu_\beta, k), \qquad f_\alpha^+(\mu_\beta, k+1) < f_\alpha^-(\mu_\beta, k),$$

$$f_\alpha^+(\mu_\beta, k), f_\alpha^-(\mu_\beta, k) \overset{C^1}{\to} 0, \qquad \text{as } k \to +\infty,$$

where $\alpha, \beta = 1, 2$, $\alpha \neq \beta$, such that the trajectories which stay forever in $\tilde{N}$ except the unstable manifold of the origin of the system E_μ are in one to one correspondence with the trajectories of the following symbolic system:

(1) $(W(0), \sigma)$, *when* $\mu_1 \leq 0, \mu_2 \leq 0$;

(2) $(V(\alpha, k), \sigma)$, *when* $f_\alpha^-(\mu_\beta, k) < \mu_\alpha < f_\alpha^-(\mu_\beta, k-1), \mu_\beta < 0$, *where $V(\alpha, k)$ is a subset of W that satisfies $W(\alpha, k+1) \subset V(\alpha, k) \subset W(\alpha, k)$; and $V(\alpha, k) = W(\alpha, k)$, when* $f_\alpha^+(\mu_\beta, k) < \mu_\alpha < f_\alpha^-(\mu_\beta, k-1), \mu_\beta < 0$;

(3) (V_k^l, σ), *when* $f_1^-(\mu_2, k) < \mu_1 < f_1^-(\mu_2, k-1), f_2^-(\mu_1, l) < \mu_2 < f_2^-(\mu_1, l-1)$, *where V_k^l is a subset of W that satisfies $W_{k+1}^{l+1} \subset V_k^l \subset W_k^l$; and $V_k^l = W_k^l$, when* $f_1^+(\mu_2, k) < \mu_1 < f_1^-(\mu_2, k-1), f_2^+(\mu_1, l) < \mu_2 < f_2^-(\mu_1, l-1)$.

The following theorem describes how the dynamic behavior of the unstable manifold of the origin changes with the variation of parameters.

THEOREM 3.2. *Under the same conditions of Theorem* 3.1, *with the same parameters* $\mu = (\mu_1, \mu_2)$ *and same families of functions* $f_\alpha^-(\mu_\beta, k)$, $f_\alpha^+(\mu_\beta, k)$ $(k \geq \tilde{k})$, *etc.*, *the dynamic behavior of the unstable manifold of the origin is described as follows*:

(1) *When* $\mu_1 < 0$, $\mu_2 < 0$, *the unstable manifold of the origin leaves* $\tilde{N}$, *the corresponding system is* Ω*-stable in* $\tilde{N}$.

(2) *When* $\mu_\beta < 0$, $\mu_\alpha = 0$, *one branch of the unstable manifold lies in the stable manifold of* $\tilde{L}_\alpha$, *the other branch which goes to* $\tilde{L}_\beta$ *leaves* $\tilde{N}$.

(3) *When* $\mu_\beta < 0$ *and* $f_\alpha^+(\mu_\beta, k+1) < \mu_\alpha < f_\alpha^-(\mu_\beta, k)$, *both branches of the unstable manifold leave* $\tilde{N}$, *the corresponding system is* Ω*-stable in* $\tilde{N}$.

(4) *When* $\mu_\beta < 0$ *and* $f_\alpha^-(\mu_\beta, k) < \mu_\alpha < f_\alpha^+(\mu_\beta, k)$, *there is an infinite number of bifurcation curves of the form* $\mu_\alpha = h(\mu_\beta)$, *which describe the type of the branch of the unstable manifold of the origin which goes to* $\tilde{L}_\alpha$ *to run around* $\tilde{L}_\alpha$ *in* $\tilde{N}$ *and whether it tends to the origin or the periodic orbit* $\tilde{L}_\alpha$ (*the corresponding orbit is a homoclinic orbit or heteroclinic orbit connecting the saddle point itself or connecting the saddle point and the periodic orbit* $\tilde{L}_\alpha$, *respectively*, *after some type of windings*), *or just runs around in* $\tilde{N}$, *not tending to any one of the three critical elements* $\tilde{L}_\gamma$ $(\gamma = 0, 1, 2)$. *For any parameter values in this region*, *the branch which goes to* $\tilde{L}_\beta$ *of the unstable manifold leaves* $\tilde{N}$. *So in this region there are three infinite families of bifurcation curves*.

(5) *When* $f_\alpha^+(\mu_\beta, k) < \mu_\alpha < f_\alpha^-(\mu_\beta, k-1)$, $f_\beta^+(\mu_\alpha, l) < \mu_\beta < f_\beta^-(\mu_\alpha, l-1)$, *both branches of the unstable manifold leave* $\tilde{N}$, *the corresponding system is* Ω*-stable in* $\tilde{N}$.

(6) *When* $f_\alpha^+(\mu_\beta, k) < \mu_\alpha < f_\alpha^-(\mu_\beta, k-1)$, $f_\beta^-(\mu_\alpha, l) < \mu_\beta < f_\beta^+(\mu_\alpha, l)$, *there are an infinite number of bifurcation curves of the form* $\mu_\beta = g(\mu_\alpha)$ *which describe the type of the branch of the unstable manifold which goes to* $\tilde{L}_\beta$ *to run around in* $\tilde{N}$ *and whether it finally tends to the origin or the periodic orbit* $\tilde{L}_1$ *or* $\tilde{L}_2$, *or just runs around in* $\tilde{N}$ *not tending to any one of the three critical elements*. *The branch which goes to* $\tilde{L}_\alpha$ *of the unstable manifold leaves* $\tilde{N}$. *Hence in this region there are four infinite families of bifurcation curves*.

(7) *When* $f_\alpha^-(\mu_\beta, k) < \mu_\alpha < f_\alpha^+(\mu_\beta, k)$, $f_\beta^-(\mu_\alpha, l) < \mu_\beta < f_\beta^+(\mu_\alpha, l)$, *there are an infinite number of bifurcation curves of the form* $\mu_\alpha = h(\mu_\beta)$, $\mu_\beta = g(\mu_\alpha)$, *respectively*, *corresponding to the various types described by some symbolic representations of the unstable manifold of the origin to run around in* $\tilde{N}$, *and whether it finally tends to the origin or the periodic orbit* $\tilde{L}_1$

or $\tilde{L}_2$, or just runs around in $\tilde{N}$ not tending to any one of the three critical elements. Hence in this region there are eight infinite families of bifurcation curves.

(8) *When $\mu_1 = \mu_2 = 0$, the branch which goes to $\tilde{L}_\alpha$ of the unstable manifold of the origin lies in the stable manifold of $\tilde{L}_\alpha$, where $\alpha = 1, 2$.*

Remark 3.1. In those parameter regions in Theorem 3.2 where the corresponding system is not Ω-stable and hence an infinite number of bifurcation curves can be found, the bifurcation curves corresponding to the unstable manifold of the origin to run around in $\tilde{N}$, not tending to any one of the three critical elements are approximated by the other two kinds of bifurcation curves arbitrarily, which are homoclinic and heteroclinic bifurcation curves, respectively.

Remark 3.2. When the parameters μ_α vary from being negative to positive, $\alpha = 1, 2$, obvious Ω-explosions happen.

Before we prove the theorems, we need to establish some lemmas.

4. SOME LEMMAS

First we study what happens if one branch of the unstable manifold of the origin, say Γ_α, stays forever in $\tilde{N}$. It follows that $\mu_\alpha = 0$ or $T_{1\alpha} M_{0\alpha} \in \sigma^0_{\alpha k}$ for some $k \geq \tilde{k}$, which implies that

$$b_\alpha^{-k}(\mu)(v_\alpha^- - \delta) \leq \mu_\alpha \leq b_\alpha^{-k}(\mu)(v_\alpha^- + \delta).$$

By the implicit function theorem, there exist C^1-functions $f_\alpha^-(\mu_\beta, k)$, $f_\alpha^+(\mu_\beta, k)$ such that the last two inequalities are equivalent to

$$f_\alpha^-(\mu_\beta, k) \leq \mu_\alpha \leq f_\alpha^+(\mu_\beta, k), \tag{9}$$

where $\alpha, \beta = 1, 2$, $\alpha \neq \beta$ and $f_\alpha^-(\mu_\beta, k), f_\alpha^+(\mu_\beta, k) \xrightarrow{C^1} 0$, when $k \to +\infty$.

The following lemmas are essential for us to discuss trajectories which stay forever in $\tilde{N}$.

LEMMA 4.1. (*a*) *If $\mu_\alpha > f_\alpha^+(\mu_\beta, k)$, then $(T_{1\alpha} T_\alpha T_{2\tilde{\alpha}} P^i_{\tilde{\alpha}} \sigma^0_{\tilde{\alpha} i}) \cap \sigma^0_{\alpha j} \neq \emptyset$, where $i \geq \tilde{k}$, $j \geq k$, $\tilde{\alpha} = 1, 2$.*

(*b*) *If $\mu_\alpha < f_\alpha^-(\mu_\beta, k)$, then $(T_{1\alpha} T_\alpha T_{2\tilde{\alpha}} P^i_{\tilde{\alpha}} \sigma^0_{\tilde{\alpha} i}) \cap \sigma^0_{\alpha j} = \emptyset$, where $i \geq \tilde{k}$, $k \geq j \geq \tilde{k}$, $\tilde{\alpha} = 1, 2$.*

Proof. It is easy to see that $P^i_{\tilde{\alpha}} \sigma^0_{\tilde{\alpha} i} = \sigma^1_{\tilde{\alpha} i}$. For any vertical curve w: $u = u_0$ in $\sigma^1_{\tilde{\alpha} i}$, $T_{2\tilde{\alpha}}$ maps it to a curve $T_{2\tilde{\alpha}} w$ in S_0^+, and according to (8), it can be expressed as

$$y = h_{\tilde{\alpha}}(u_0, z, \mu), \tag{10}$$

with $|z| \leq \delta_1$ (a constant).

One part of this curve is mapped by $T_{1\alpha}T_\alpha$ to a curve $\overline{w}$ in Λ_α with the expressions

$$\begin{aligned} u &= u_\alpha^+ + u_\alpha(\mu) + J_\alpha(\mu)\phi_\alpha(h_{\tilde{\alpha}}, z, \mu)|z|^{\theta_1(\mu)} + \cdots, \\ v &= \mu_\alpha + K_\alpha(\mu)\phi_\alpha(h_{\tilde{\alpha}}, z, \mu)|z|^{\theta_1(\mu)} + \cdots, \end{aligned} \tag{11}$$

where the dots denote the higher-order terms. Let $|z|^{\theta_1(\mu)} = (-1)^{\alpha-1}p$, and notice $C_\alpha < 0$, by using the implicit function theorem, p is expressed as a function $\tilde{V}_\alpha$ of v, with the parameters u_0, μ, $\tilde{\alpha}, i$,

$$p = \tilde{V}_\alpha(u_0, v, \tilde{\alpha}, i, \mu), \tag{12}$$

where $v \in [\mu_\alpha - \delta_2, \mu_\alpha]$ (δ_2 is a positive constant). Combining (12) with the first equation of (11), we know $T_{1\alpha}T_\alpha T_{2\tilde{\alpha}}w$ can be expressed as

$$u = \tilde{H}_\alpha(u_0, v, \tilde{\alpha}, i, \mu), \tag{13}$$

where $\tilde{H}_\alpha \in C^1, v \in [\mu_\alpha - \delta_2, \mu_\alpha], \tilde{H}_\alpha(u_0, \mu_\alpha, \tilde{\alpha}, i, \mu) = u_\alpha^+ + u_\alpha(\mu)$. It follows that for any horizontal curve w_1: $v = v_1$ in $\sigma_{\alpha j}^0$ $(j \geq k)$, $T_{1\alpha}T_\alpha T_{2\tilde{\alpha}}w \cap w_1 \neq \emptyset$, since $\mu_\alpha > f_\alpha^+(\mu_\beta, k)$ implies that $\mu_\alpha > b_\alpha^{-k}(\mu)(v_\alpha^- + \delta)$, where we should notice that $k \geq \tilde{k}, \tilde{k}$ large enough and δ_2 is a constant which does not depend on the number k and we just consider our question in a sufficiently small neighborhood of the origin in the (μ_1, μ_2) parameter plane. In this way, the conclusion (a) is proved.

Conclusion (b) is proved similarly.

In order to study whether or not a trajectory in $\tilde{N}$ will tend to $L_\alpha(\mu)$ when $t \to +\infty$, that is, if it lies in the stable manifold of this periodic orbit, we need another lemma.

Now we introduce some more notations. Let

$$\sigma_\alpha = \{(u, v) \mid (u, v) \in U_\alpha, |u| \leq 1, |v| \leq 1\}.$$

Consider the map P_α^k: $\sigma_\alpha \to \sigma_\alpha, (u, v) \mapsto (\overline{u}, \overline{v})$, which can also be written as $\overline{u} = a_\alpha^k(\mu)u$, $v = b_\alpha^{-k}(\mu)\overline{v}$. The domain $\overline{\sigma}_{\alpha k}^0$ and the range $\overline{\sigma}_{\alpha k}^1$ of the above map P_α^k: $\sigma_\alpha \to \sigma_\alpha$ are as follows:

$$\begin{aligned} \overline{\sigma}_{\alpha k}^0 &= \{(u, v) \mid |u| \leq 1, |v| \leq b_\alpha^{-k}(\mu)\}, \\ \overline{\sigma}_{\alpha k}^1 &= \{(u, v) \mid |u| \leq a_\alpha^k(\mu), |v| \leq 1\}. \end{aligned}$$

LEMMA 4.2. (a) *If* $\mu_\alpha > 0$, *then* $(T_{1\alpha}T_\alpha T_{2\tilde{\alpha}}P_{\tilde{\alpha}}^i \sigma_{\tilde{\alpha} i}^0) \cap \overline{\sigma}_{\alpha k}^0 \neq \emptyset$, *where* $i \geq \tilde{k}$, *k large enough*, $\tilde{\alpha} = 1, 2$.

(b) *If* $\mu_\alpha \leq 0$, *there is no trajectory except the branch* Γ_α *of the unstable manifold of the origin which may stay forever in* $\tilde{N}$ *and run around* *the contour of the cycles and finally tends to the periodic orbit* $\tilde{L}_\alpha$ *when* $t \to +\infty$.

Proof. The proof of conclusion (a) is similar to Lemma 4.1.

Now we prove conclusion (b). According to the second equation of (11), under the map $T_{1\alpha}T_\alpha$, the expression of the v-coordinate is $v = \mu_\alpha + K_\alpha(\mu)\phi_\alpha(h_{\tilde{\alpha}}, z, \mu)|z|^{\theta_1(\mu)} + \cdots$. Since for the trajectory which runs around in $\tilde{N}$, not tending to the origin, we always have $|z| > 0$, so $\mu_\alpha \leq 0$ implies that $v < 0$. It follows that there is no trajectory in $\tilde{N}$ which runs around and finally tends to the periodic orbit $\tilde{L}_\alpha$.

Second, we introduce a fixed-point theorem in direct product of spaces [2, 14], which will be used in the proofs of the theorems. Suppose that we are given two sequences of complete metric spaces X_i and Y_i with metrics d_i and ρ_i ($i = 0, \pm 1, \pm 2, \dots$), respectively, which satisfy

$$\sup_{x_i^1, x_i^2 \in X_i} d_i\left(x_i^1, x_i^2\right) < \delta, \qquad \sup_{y_i^1, y_i^2 \in Y_i} \rho_i\left(y_i^1, y_i^2\right) < \delta,$$

where δ is any fixed constant. And there are operators A_i: $X_i \times Y_{i+1} \to X_{i+1}$, B_{i+1}: $X_i \times Y_{i+1} \to Y_i$, which satisfy

$$d_{i+1}\left(\bar{x}_{i+1}^1, \bar{x}_{i+1}^2\right) < q\left(d_i\left(x_i^1, x_i^2\right) + \rho_{i+1}\left(y_{i+1}^1, y_{i+1}^2\right)\right)/2,$$
$$\rho_i\left(\bar{y}_i^1, \bar{y}_i^2\right) < q\left(d_i\left(x_i^1, x_i^2\right) + \rho_{i+1}\left(y_{i+1}^1, y_{i+1}^2\right)\right)/2,$$

where $\bar{x}_{i+1}^j = A_i(x_i^j, y_{i+1}^j)$, $\bar{y}_i^j = B_{i+1}(x_i^j, y_{i+1}^j)$, $j = 1, 2$, q a constant.

Obviously, $Z = \prod_{-\infty}^{\infty} (X_i \times Y_i)$ is a complete metric space with the metric d,

$$d\left(\left(x^1, y^1\right), \left(x^2, y^2\right)\right) = \sup_i d_i\left(x_i^1, x_i^2\right) + \sup_i \rho_i\left(y_i^1, y_i^2\right),$$

where $z = (x, y) = (\dots, (x_{-1}, y_{-1}), (x_0, y_0), (x_1, y_1), \dots)$. We define an operator C on Z as

$$z = (x, y) \xrightarrow{C} \bar{z} = (\bar{x}, \bar{y}),$$

where $\bar{x}_{i+1} = A_i(x_i, y_{i+1})$, $\bar{y}_i = B_{i+1}(x_i, y_{i+1})$.

LEMMA 4.3. *The operator* C *is a contraction map on* Z *when* $q < 1$.

For the proof of this lemma, see [14].

Let $z^* = (x^*, y^*)$ be the fixed point of the operator C. Expanded in coordinates, this point is

$$\left(\dots, \left(x_{-1}^*, y_{-1}^*\right), \left(x_0^*, y_0^*\right), \left(x_1^*, y_1^*\right), \dots\right),$$

satisfying the conditions

$$x_{i+1}^{*} = A_i(x_i^{*}, y_{i+1}^{*}),\ y_i^{*} = B_{i+1}(x_i^{*}, y_{i+1}^{*}).$$

5. PROOF OF THEOREMS

Now we show how these lemmas apply to the discussions. For example, if the condition (a) of Lemma 4.1 holds, we have

$$\left(T_{1\alpha} T_{\alpha} T_{2\tilde{\alpha}} P_{\tilde{\alpha}}^{i} \sigma_{\tilde{\alpha}i}^{0}\right) \cap \sigma_{\alpha j}^{0} \neq \varnothing.$$

Hence we have a map

$$T_{1\alpha} T_{\alpha} T_{2\tilde{\alpha}} P_{\tilde{\alpha}}^{i}\colon \sigma_{\tilde{\alpha}i}^{0} \to \sigma_{\alpha j}^{0}, \qquad (u_0, v_0) \mapsto (\bar{u}_0, \bar{v}_0),$$

$$\sigma_{\tilde{\alpha}i}^{0} \xrightarrow{P_{\tilde{\alpha}}^{i}} \sigma_{\tilde{\alpha}i}^{1} \xrightarrow{T_{2\tilde{\alpha}}} S_0^{+} \xrightarrow{T_{1\alpha}T_{\alpha}} \sigma_{\alpha j}^{0} \xrightarrow{P_{\alpha}^{j}} \sigma_{\alpha j}^{1} \xrightarrow{T_{2\alpha}} S_0^{+}, \tag{14}$$

$$(u_0, v_0) \mapsto (u_1, v_1) \mapsto (y, z) \mapsto (\bar{u}_0, \bar{v}_0) \mapsto (\bar{u}_1, \bar{v}_1) \mapsto (\bar{y}, \bar{z}).$$

Since the map $P_{\tilde{\alpha}}^{i}\colon \sigma_{\tilde{\alpha}i}^{0} \to \sigma_{\tilde{\alpha}i}^{1}$ can be written as $u_1 = a_{\tilde{\alpha}}^{i}(\mu) u_0 = \xi_{\tilde{\alpha}}(u_0, i, \mu)$, $v_0 = b_{\tilde{\alpha}}^{-i}(\mu) v_1 = \eta_{\tilde{\alpha}}(v_1, i, \mu)$, the map $T_{2\tilde{\alpha}} P_{\tilde{\alpha}}^{i}\colon \sigma_{\tilde{\alpha}i}^{0} \to S_0^{+}$ has the form

$$\begin{aligned} y &= f_{\tilde{\alpha}}(\xi_{\tilde{\alpha}}(u_0, i, \mu), v_1, \mu), \\ z &= g_{\tilde{\alpha}}(\xi_{\tilde{\alpha}}(u_0, i, \mu), v_1, \mu). \end{aligned} \tag{15}$$

Noticing (8), we can change Eq. (15) into

$$\begin{aligned} y &= Y(u_0, z, \tilde{\alpha}, i, \mu), \\ v_1 &= V(u_0, z, \tilde{\alpha}, i, \mu). \end{aligned} \tag{16}$$

In terms of the variables (u_0, z), the map $T_{1\alpha} T_{\alpha} T_{2\tilde{\alpha}} P_{\tilde{\alpha}}^{i}\colon \sigma_{\tilde{\alpha}i}^{0} \to \sigma_{\alpha j}^{0}$ has the form

$$\begin{aligned} \bar{u}_0 &= U_{\alpha}\left(\phi_{\alpha}(Y(u_0, z, \tilde{\alpha}, i, \mu), z, \mu)|z|^{\theta_1(\mu)}, \mu\right), \\ &\eta_{\alpha}(V(\bar{u}_0, \bar{z}, \alpha, j, \mu), j, \mu) \\ &\quad = V_{\alpha}\left(\phi_{\alpha}(Y(u_0, z, \tilde{\alpha}, i, \mu), z, \mu)|z|^{\theta_1(\mu)}, \mu\right). \end{aligned} \tag{17}$$

After a variable transformation $z = \operatorname{sgn}(p)|p|^{1/\theta_1} = f(p)$, $\bar{z} = \operatorname{sgn}(\bar{p})|\bar{p}|^{1/\theta_1} = f(\bar{p})$, (17) is transformed into

$$\begin{aligned} \bar{u}_0 &= U_{\alpha}\left(\phi_{\alpha}(Y(u_0, f(p), \tilde{\alpha}, i, \mu), f(p), \mu)|p|, \mu\right), \\ \bar{\eta}_{\alpha}(\bar{u}_0, \bar{p}, j, \mu) &\triangleq \eta_{\alpha}(V(\bar{u}_0, f(\bar{p}), \alpha, j, \mu), j, \mu) \\ &= V_{\alpha}\left(\phi_{\alpha}(Y(u_0, f(p), \tilde{\alpha}, i, \mu), f(p), \mu)|p|, \mu\right). \end{aligned} \tag{18}$$

Let $\overline{\phi}_\alpha = (-1)^{\alpha-1}\phi_\alpha$. Then the total differentiation of (18) gives

$$d\bar{u}_0 = \frac{\partial U_\alpha}{\partial(x,y)}\left(\frac{\partial\overline{\phi}_\alpha}{\partial u_0}p\,du_0 + \frac{\partial\overline{\phi}_\alpha}{\partial p}p\,dp + \overline{\phi}_\alpha\,dp\right),$$

$$\frac{\partial\overline{\eta}_\alpha}{\partial\bar{u}_0}\,d\bar{u}_0 + \frac{\partial\overline{\eta}_\alpha}{\partial\bar{p}}\,d\bar{p} = \frac{\partial V_\alpha}{\partial(x,y)}\left(\frac{\partial\overline{\phi}_\alpha}{\partial u_0}p\,du_0 + \frac{\partial\overline{\phi}_\alpha}{\partial p}p\,dp + \overline{\phi}_\alpha\,dp\right).$$

So if $p, \mu \to 0$ and $i, j \to +\infty$, we have

$$\frac{\partial\overline{\phi}_\alpha}{\partial u_0}p,\ \frac{\partial\overline{\phi}_\alpha}{\partial p}p,\ \frac{\partial\overline{\eta}_\alpha}{\partial\bar{u}_0},\ \frac{\partial\overline{\eta}_\alpha}{\partial\bar{p}} \to 0, \qquad \frac{\partial V_\alpha}{\partial(x,y)}\overline{\phi}_\alpha \to (-1)^{\alpha-1}C_\alpha \neq 0,$$

where the estimates of derivatives in Lemma 2.1, the chain rule, and the condition $0 < \theta_1(\mu) < 1$ are used. By the implicit function theorem, Eq. (18) can be written as

$$\bar{u}_0 = \tilde{U}(u_0, \bar{p}, \tilde{\alpha}, i, \alpha, j, \mu), \qquad p = \tilde{P}(u_0, \bar{p}, \tilde{\alpha}, i, \alpha, j, \mu), \tag{19}$$

where

$$\left|\frac{\partial\tilde{U}}{\partial u_0}\right| + \left|\frac{\partial\tilde{U}}{\partial\bar{p}}\right| + \left|\frac{\partial\tilde{P}}{\partial u_0}\right| + \left|\frac{\partial\tilde{P}}{\partial\bar{p}}\right| \ll 1. \tag{20}$$

The domain of definition of this map is $\{|u_0 - u^+_{\tilde{\alpha}}| \leq \delta, |\bar{p}| \leq \delta_3\}$, where δ_3 is a sufficiently small constant, and the range is in the set $\{|\bar{u}_0 - u^+_\alpha| \leq \delta, |p| \leq \delta_3\}$.

Now, as an illustration, we make a discussion for the case

$$f_1^+(\mu_2, k) < \mu_1 < f_1^-(\mu_2, k-1), \qquad f_2^+(\mu_1, l) < \mu_2 < f_2^-(\mu_1, l-1).$$

We show that the existence of a trajectory which runs around in $\tilde{N}$, not tending to any one of $\tilde{L}_\gamma$ $(\gamma = 0,1,2)$ is equivalent to the existence of a fixed point of the following operator on a product space:

Let $X = \{(u_0^i, p^i)_{i\in\mathbf{Z}}\,|\,|u_0^i - u^+_{\alpha_i}| \leq \delta, |p| \leq \delta_3\}$, and define an operator S: $X \to X, (u_0^i, v^i)_{i\in\mathbf{Z}} \mapsto (\bar{u}_0^i, \bar{v}^i)_{i\in\mathbf{Z}}$ by the family of maps

$$\begin{aligned} A_i&: \bar{u}_0^{i+1} = \tilde{U}(u_0^i, p^{i+1}, \alpha_i, k_i, \alpha_{i+1}, k_{i+1}, \mu),\\ B_{i+1}&: \bar{p}_0^i = \tilde{P}(u_0^i, p^{i+1}, \alpha_i, k_i, \alpha_{i+1}, k_{i+1}, \mu),\\ &\qquad i = 0, \pm1, \pm2, \dots; \end{aligned} \tag{21}$$

here the admissible symbol sequence

$$(\dots, (\alpha_{-i}, k_{-i}), \dots, (\alpha_0, k_0), \dots, (\alpha_i, k_i), \dots) \tag{22}$$

satisfies

$$k_i \in \mathbf{N}, k_i \geq k \text{ if } \alpha_i = 1, \qquad k_i \geq l \text{ if } \alpha_i = 2. \tag{23}$$

Then by the fixed-point theorem of product spaces (Lemma 4.3) and the estimates in (20), it follows S: $X \to X$ has a unique fixed point corresponding to the admissible symbol sequence (22). Hence there is a unique point in $\sigma^0_{\alpha_0 k_0}$, such that the trajectory which passes through this point stays forever in $\tilde{N}$ and intersects with $\sigma^0_{\alpha_i k_i}$ successively. Moreover, by Lemma 4.1, if $f_1^+(\mu_2, k) < \mu_1 < f_1^-(\mu_2, k-1), f_2^+(\mu_1, l) < \mu_2 < f_2^-(\mu_1, l-1)$, where $k \geq \tilde{k} + 1, l \geq \tilde{k} + 1$, then the winding type of a trajectory which stays forever in $\tilde{N}$ described by some symbol sequence should satisfy the condition (23), that is, it must be admissible.

Similarly we can discuss the existence of trajectories which tend to the singular point (the origin) or the periodic orbits and establish their correspondence with some admissible symbol sequences. In the process of the above discussions some techniques developed in [2] are used. For example, in the process when we discuss the trajectory which lies in the unstable manifold of $\tilde{L}_\beta$, and in the stable manifold of $\tilde{L}_\alpha$, i.e., trajectory in $W^u_\beta \cap W^s_\alpha$, whose symbolic description is in $\Omega_{\beta\alpha}$, we will have to make use of Lemma 4.2; for details, consult [2, 3]. In this way, we obtain the symbolic description of the trajectories except the unstable manifold of the origin which stays forever in the neighborhood of $\tilde{N}$ of the contour, and Theorem 3.1 is proved.

Now we begin to study the changes of dynamic behavior of the unstable manifold of the origin with respect to variations of parameters and study whether it is possible for the unstable manifold of the origin to run around in $\tilde{N}$, not going away, under the condition

$$f_\alpha^-(\mu_\beta, k) < \mu_\alpha < f_\alpha^+(\mu_\beta, k), \qquad f_\beta^+(\mu_\alpha, l+1) < \mu_\beta < f_\beta^-(\mu_\alpha, l). \tag{24}$$

It is obvious that the branch of the unstable manifold which goes to $\tilde{L}_\beta$ finally leaves $\tilde{N}$ for parameters in the region (24).

We discuss whether it is possible for the branch of the unstable manifold which goes to $\tilde{L}_\alpha$ to run around in $\tilde{N}$, not tending to any one of $\tilde{L}_\gamma$ $(\gamma = 0, 1, 2)$.

At this time a symbol sequence

$$((\alpha, k), (\alpha_1, k_1), (\alpha_2, k_2), \ldots, (\alpha_i, k_i), \ldots) = s \tag{25}$$

is said to be admissible if $k_i \in \mathbf{N}$, $k_i \geq k + 1$ when $\alpha_i = \alpha$ and $k_i \geq l + 1$ when $\alpha_i = \beta$.

We show that for any admissible symbol sequence (25), there exists a unique parameter curve $\mu_\alpha = h(\mu_\beta)$ in the region (24) such that for parameters in this curve the branch which goes to $\tilde{L}_\alpha$ of the unstable manifold of the origin runs around in $\tilde{N}$ with the winding type $((\alpha, k), (\alpha_1, k_1), \dots, (\alpha_i, k_i), \dots)$, not tending to any one of $\tilde{L}_\gamma$ $(\gamma = 0, 1, 2)$. Here (α_i, k_i) means that the trajectory runs along the periodic orbit $\tilde{L}_{\alpha_i}$ for k_i times.

Let us show how to obtain this result. We define

$$X_1 = \left\{ \left(u_0^1, p^1 \right), \left(u_0^2, p^2 \right), \dots \right) \mid |u_0^i - u_{\alpha_i}^+| \le \delta, |p^i| \le \delta_3 \right\},$$

and consider the operator S_1: $X_1 \to X_1$, $(u_0^i, p^i)_{i \in \mathbf{Z}_+} \mapsto (\bar{u}_0^i, \bar{p}^i)_{i \in \mathbf{Z}_+}$ defined by the family of maps

$$\begin{aligned} A_i &: \bar{u}_0^{i+1} = \tilde{U}\left(u_0^i, p^{i+1}, \alpha_i, k_i, \alpha_{i+1}, k_{i+1}, \mu \right), \qquad i = 0, 1, 2, \dots, \\ B_{i+1} &: \bar{p}^i = \tilde{P}\left(u_0^i, p^{i+1}, \alpha_i, k_i, \alpha_{i+1}, k_{i+1}, \mu \right), \qquad i = 1, 2, \dots, \end{aligned}$$

with u_0^0 being a parameter, $|u_0^0 - u_\alpha^+| \le \delta$, $(\alpha_0, k_0) = (\alpha, k)$. Hence the operator S_1 depends on (u_0^0, μ).

Noticing the process of deriving (19), we know that S_1 in fact C^1 depends on (u_0^0, μ). It follows that S_1: $X_1 \to X_1$ has a unique fixed point $(u_0^i(u_0^0, \mu), \mu)_{i \in \mathbf{Z}_+}$ by Lemma 4.3, and this point depends smoothly on the parameters (u_0^0, μ) in the sense of C^1-metric. Let $p^0 = \tilde{P}(u_0^0, p^1(u_0^0, \mu), \alpha, k, \alpha_1, k_1, \mu)$. Then (u_0^0, μ) determines a unique point in $\sigma_{\alpha k}^0$ such that the trajectory which passes through this point travel in $\tilde{N}$ with the winding type $((\alpha, k), (\alpha_1, k_1), (\alpha_2, k_2), \dots)$, and when u_0^0 changes in the region $|u_0^0 - u_\alpha^+| \le \delta$, the trace of these points is a differentiable curve

$$v_0^0 = \eta_\alpha\left(u_0^0, v_1^0\left(u_0^0, \mu \right), k, \mu \right) \tag{26}$$

in $\sigma_{\alpha k}^0$, where $v_1^0(u_0^0, \mu)$ is a differentiable function determined by $p^0(u_0^0, \mu)$ with respect to Eq. (16). Then we can solve the equation

$$\begin{aligned} \mu_\alpha &= \eta_\alpha\left(u_\alpha^+ + u_\alpha(\mu), v_1^0\left(u_\alpha^+ + u_\alpha(\mu), \mu \right), k, \mu \right) \\ &= b_\alpha^{-k}(\mu) v_1^0\left(u_\alpha^+ + u_\alpha(\mu), \mu \right) \end{aligned} \tag{27}$$

and obtain the parameter curve $\mu_\alpha = h_s(\mu_\beta)$.

Similarly, by considering the other types of admissible symbol sequences

$$\left((\alpha, k), (\alpha_1, k_1), \dots, (\alpha_i, k_i), (\gamma, \infty), \dots, (\gamma, \infty), \dots \right) = s_\gamma, \tag{28}$$

where $\gamma = 0$, 1, or 2, with the requirements that

$$k_i \in \mathbf{N} \text{ and } k_i \geq k + 1 \text{ if } \alpha_i = \alpha, \qquad k_i \geq l + 1 \text{ if } \alpha_i = \beta, \tag{29}$$

we obtain three kinds of infinite number of bifurcation curves of the form $\mu_\alpha = h_{s_\gamma}(\mu_\beta)$, where $\gamma = 0$, 1, or 2; s_γ satisfies the admissibility condition (29). When the parameter is in one of these curves, the branch which goes to $\tilde{L}_\alpha$ of the unstable manifold of the origin finally lies in the local stable manifold of the critical elements $\tilde{L}_\gamma$ $(\gamma = 0, 1, 2)$ after the s_γ-type of windings. Hence if $\gamma = 0$, $\tilde{L}_0$ is the origin, the branch of the unstable manifold of the origin is an s_0-type homoclinic orbit; if $\gamma = 1, 2$, $\tilde{L}_\gamma$ is the periodic orbit and the branch is an s_γ-type heteroclinic orbit.

The fact that the first kind of parameter curve is a bifurcation curve is proved by showing that this kind of parameter curve is approximated by the homoclinic bifurcation curves and heteroclinic bifurcation curves with the winding types satisfying the admissibility condition (29). For details, see [3]. Hence in this parameter region there are four kinds of bifurcation curves.

In the region $f_\alpha^-(\mu_\beta, k) < \mu_\alpha < f_\alpha^+(\mu_\beta, k), f_\beta^-(\mu_\alpha, l) < \mu_\beta < f_\beta^+(\mu_\alpha, l)$, the situation is different from that in the region (24) in that here it is possible for the branch of the unstable manifold of the origin which goes to $\tilde{L}_\beta$ to run around in $\tilde{N}$, not going away. The process of obtaining bifurcation curves is similar to the above discussions and eight kinds of infinite number of bifurcation curves can be found in this parameter region.

According to the requirements of admissible sequences, when k or l varies, the form of admissible sequences is different. And we can apply these kinds of discussions to other parameter regions to obtain all the conclusions of the theorem. For the region $\mu_\beta < 0$, $f_\alpha^-(\mu_\beta, k) < \mu_\alpha < f_\alpha^+(\mu_\beta, k)$ or $\mu_\beta < 0, f_\alpha^+(\mu_\beta, k+1) < \mu_\alpha < f_\alpha^-(\mu_\beta, k)$, the statements of Lemma 4.1 and 4.2 are very essential in the discussions. In this way, we can prove Theorem 3.2.

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